

Fractional kinetic hierarchies and intermittency

Anatoly N. Kochubei

Institute of Mathematics,
National Academy of Sciences of Ukraine,
Tereshchenkivska 3,
Kyiv, 01601 Ukraine
Email: kochubei@imath.kiev.ua

Yuri Kondratiev

Department of Mathematics, University of Bielefeld,
D-33615 Bielefeld, Germany,
Email: kondrat@math.uni-bielefeld.de

Abstract

We consider general convolutional derivatives and related fractional statistical dynamics of continuous interacting particle systems. We apply the subordination principle to construct kinetic fractional statistical dynamics in the continuum in terms of solutions to Vlasov-type hierarchies. Conditions for the intermittency property of fractional kinetic dynamics are obtained.

Keywords Statistical dynamics, generalized fractional derivatives, Vlasov-type scaling limit, correlation functions, Poisson flow, intermittency

1 Introduction

Kinetic equations for classical gases may be derived from the BBGKY hierarchies for time dependent correlation functions which describe Hamiltonian dynamics of gases, see e.g. an excellent review by H.Spohn [38]. Making scalings in BBGKY hierarchical chains, we will arrive in the limiting kinetic hierarchies of Boltzmann or Vlasov type depending on the particular scaling we use. Both kinetic hierarchies have a common chaos propagation property. Using this property we obtain Boltzmann or Vlasov equation respectively as non-linear equations for the density of the considered system.

A similar approach may be also applied to Markov dynamics of interacting particle systems in the continuum as it was proposed in [15]. These dynamics may be described on the microscopic level by means of related hierarchical evolution equations for correlation functions and proper scalings will lead to limiting mesoscopic hierarchies and corresponding kinetic equations. Again, a common point for resulting hierarchies is the chaos propagation property that is a root of the kinetic equation for the density of the system. Note that this property means that

the kinetic state evolution of the system will be given by a flow of Poisson measures provided the initial state is a Poisson measure. Of course, a rigorous realization of this scheme (that includes such steps as construction of the microscopic Markov dynamics, control of the convergence of solutions for rescaled evolutions and an analysis of corresponding kinetic equations) shall be done for each particular model and is, in general, quite difficult technical problem. At the present time, this program is realized for a number of Markov dynamics of continuous systems which includes certain birth-and-death processes, Kawasaki type dynamics, binary jumps models, see e.g. [15, 16, 18].

In the present paper we extend described above approach to the case of certain non-Markov dynamics of interacting particle systems in the continuum. Namely, we will consider hierarchical evolution equations for correlation functions with general fractional time derivatives of convolutional type. From the stochastic point of view, the latter corresponds to a random time change in the original Markov processes and effectively leads to a memory effect in stochastic dynamics. The Vlasov type mesoscopic scaling for the fractional hierarchical chains will affect only spatial structure of their generators and will give kinetic hierarchies of the same form as before but with fractional time derivatives. The latter drastically change the structure of their solutions. In terms of corresponding state evolutions we obtain subordinations of Poisson flows. The latter means that in the fractional case the kinetic hierarchies are not reduced just to density evolutions. Time development of correlation functions in such hierarchical chains is essentially different for all levels of the hierarchy. In other words, the kinetic description of the dynamics needs to work with all the hierarchy but not only with the evolution of the density. As a very prominent effect of this situation we will show an intermittency property for certain classes of fractional kinetic dynamics. This property means a progressive growth in the time for correlation functions of higher orders and never may be observed for Poisson flows. Note that for the classical case of Caputo-Djrbashian fractional derivatives we pointed out this effect in our previous paper [10]. Here we are dealing with a large class of generalized time derivatives including, in particular, the case of so-called distributed order derivatives.

2 General fractional calculus

2.1. In order to explain a general concept of fractional calculus developed in [25], we need some notions from function theory connected with properties of the Laplace transform. For their detailed exposition see [37].

A real-valued function f on $(0, \infty)$ is called a *Bernstein function*, if $f \in C^\infty$, $f(\lambda) \geq 0$ for all $\lambda > 0$, and

$$(-1)^{n-1} f^{(n)}(\lambda) \geq 0 \quad \text{for all } n \geq 1, \lambda > 0,$$

so that its derivative $g = f'$ is completely monotone, that is $(-1)^m g^{(m)}(\lambda) \geq 0$, $m = 0, 1, 2, \dots$

Equivalently, a function $f : (0, \infty) \rightarrow \mathbb{R}$ is a Bernstein function, if and only if

$$f(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t}) \mu(dt) \tag{2.1}$$

where $a, b \geq 0$, and μ is a Borel measure on $[0, \infty)$, called the *Lévy measure*, such that

$$\int_0^\infty \min(1, t) \mu(dt) < \infty. \quad (2.2)$$

The triplet (a, b, μ) is determined by f uniquely. In particular,

$$a = f(0+), \quad b = \lim_{\lambda \rightarrow \infty} \frac{f(\lambda)}{\lambda}. \quad (2.3)$$

A Bernstein function f is said to be a *complete Bernstein function*, if its Lévy measure μ has a completely monotone density $m(t)$ with respect to the Lebesgue measure, so that (2.1) takes the form

$$f(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t}) m(t) dt \quad (2.4)$$

where, by (2.2),

$$\int_0^\infty \min(1, t) m(t) dt < \infty.$$

Here the complete monotonicity means that $m \in C^\infty(0, \infty)$, $(-1)^n m^{(n)}(t) \geq 0$, $t > 0$, for all $n = 0, 1, 2, \dots$

Another important class of functions is that of *Stieltjes functions*, that is of functions φ admitting the integral representation

$$\varphi(\lambda) = \frac{a}{\lambda} + b + \int_0^\infty \frac{1}{\lambda + t} \sigma(dt) \quad (2.5)$$

where $a, b \geq 0$, σ is a Borel measure on $[0, \infty)$, such that

$$\int_0^\infty (1 + t)^{-1} \sigma(dt) < \infty. \quad (2.6)$$

Using the identity $(\lambda + t)^{-1} = \int_0^\infty e^{-ts} e^{-\lambda s} ds$ we find from (2.5) that

$$\varphi(\lambda) = \frac{a}{\lambda} + b + \int_0^\infty e^{-\lambda s} g(s) ds \quad (2.7)$$

where

$$g(s) = \int_0^\infty e^{-ts} \sigma(dt) \quad (2.8)$$

is a completely monotone function whose Laplace transform exists for any $\lambda > 0$.

We will denote the class of complete Bernstein functions by \mathcal{CBF} , and the class of Stieltjes functions by \mathcal{S} . The following characterization is proved in [37]: for a nonnegative function f on $(0, \infty)$, the following conditions are equivalent.

(i) $f \in \mathcal{CBF}$.

(ii) The function $\lambda \mapsto \lambda^{-1}f(\lambda)$ is in \mathcal{S} .

(iii) f has an analytic continuation to the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$, such that $\text{Im } f(z) \geq 0$ for all $z \in \mathbb{H}$, and there exists the real limit

$$f(0+) = \lim_{(0,\infty) \ni \lambda \rightarrow 0} f(\lambda). \quad (2.9)$$

(iv) f has an analytic continuation to the cut complex plane $\mathbb{C} \setminus (-\infty, 0]$, such that $\text{Im } z \cdot \text{Im } f(z) \geq 0$, and there exists the real limit (2.9).

(v) f has an analytic continuation to \mathbb{H} given by the expression

$$f(z) = a + bz + \int_0^\infty \frac{z}{z+t} \sigma(dt) \quad (2.10)$$

where $a, b \geq 0$, and σ is a Borel measure on $(0, \infty)$ satisfying (2.6).

Note that the constants a, b are the same in both the representations (2.4) and (2.10). The density $m(t)$ appearing in the integral representation (2.4) of a function $f \in \mathcal{CBF}$ and the measure σ corresponding to the Stieltjes function $\varphi(\lambda) = \lambda^{-1}f(\lambda)$ are connected by the relation

$$m(t) = \int_0^\infty e^{-ts} s \sigma(ds).$$

The importance of complete Bernstein functions is caused by the following “nonlinear” properties [37] having significant applications.

Proposition 2.1. (i) *A function $f \neq 0$ is a complete Bernstein function, if and only if $1/f$ is a Stieltjes function.*

(ii) *Let $f, f_1, f_2 \in \mathcal{CBF}$, $\varphi, \varphi_1, \varphi_2 \in \mathcal{S}$. Then $f \circ \varphi \in \mathcal{S}$, $\varphi \circ f \in \mathcal{S}$, $f_1 \circ f_2 \in \mathcal{CBF}$, $\varphi_1 \circ \varphi_2 \in \mathcal{CBF}$, $(\lambda + f)^{-1} \in \mathcal{S}$ for any $\lambda > 0$.*

2.2. In fractional evolution equations, instead of the first time derivative, one considers nonlocal integro-differential operators. The simplest example of such an operator, for which a well-posed Cauchy problem is formulated as for the first order equations, is the Caputo-Djrbashian fractional derivative

$$(\mathbb{D}^{(\alpha)}u)(t) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} u(\tau) d\tau - t^{-\alpha} u(0) \right], \quad t > 0, \quad (2.11)$$

where $0 < \alpha < 1$. For further details see, for example, [22].

More generally, it is natural to consider differential-convolution operators

$$(\mathbb{D}_{(k)}u)(t) = \frac{d}{dt} \int_0^t k(t-\tau)u(\tau) d\tau - k(t)u(0) \quad (2.12)$$

where $k \in L_1^{\text{loc}}(\mathbb{R}_+)$ is a nonnegative function.

A nontrivial example of an operator (2.12) is a distributed order derivative $\mathbb{D}^{(\mu)}$ corresponding to

$$k(t) = \int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \mu(\alpha) d\alpha, \quad t > 0, \quad (2.13)$$

with a continuous weight function μ ; a further generalization deals with the integration with respect to a Borel measure [9, 23, 24].

Evolution equations with the fractional derivative (2.11) are widely used in physics [31, 32, 30] for modeling slow relaxation and diffusion processes; in the latter, a power-like decay of the mean square displacement of a diffusive particle appears instead of the classical exponential decay. Equations with the distributed order operators (2.12)-(2.13) describe ultraslow processes with logarithmic decay.

Considering a general operator (2.12), it is natural to ask the following question. Under what conditions upon a nonnegative function $k \in L_1^{\text{loc}}(\mathbb{R}_+)$ does the operator $\mathbb{D}_{(k)}$ possess a right inverse (a kind of a fractional integral) and produce, as a kind of a fractional derivative, equations of evolution type? The latter means, in particular, that

(A) The Cauchy problem

$$(\mathbb{D}_{(k)}u)(t) = -\lambda u(t), \quad t > 0; \quad u(0) = 1, \quad (2.14)$$

where $\lambda > 0$, has a unique solution u_λ , infinitely differentiable for $t > 0$ and completely monotone, that is $(-1)^n u_\lambda^{(n)}(t) \geq 0$ for all $t > 0$, $n = 0, 1, 2, \dots$

(B) The Cauchy problem

$$(\mathbb{D}_{(k)}w)(t, x) = \Delta w(t, x), \quad t > 0, \quad x \in \mathbb{R}^n; \quad w(0, x) = w_0(x), \quad (2.15)$$

where w_0 is a bounded globally Hölder continuous function, that is $|w_0(\xi) - w_0(\eta)| \leq C|\xi - \eta|^\gamma$, $0 < \gamma \leq 1$, for any $\xi, \eta \in \mathbb{R}^n$, has a unique bounded solution (the notion of a solution should be defined appropriately). Moreover, the equation in (2.15) possesses a fundamental solution of the Cauchy problem, a kernel with the property of a probability density.

Note that the well-posedness of the Cauchy problem for equations with the operator $\mathbb{D}_{(k)}$ has been established under much weaker assumptions than those needed for (A) and (B); see [20].

In the above special cases (A) and (B) are satisfied; see [13, 23]. When $\mathbb{D}_{(k)}$ is the Caputo-Djrbashian fractional derivative $\mathbb{D}^{(\alpha)}$, $0 < \alpha < 1$, then $u_\lambda(t) = E_\alpha(-\lambda t^\alpha)$ where E_α is the Mittag-Leffler function:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1 + \alpha n)}.$$

It is important to note some asymptotic properties of E_α for real arguments [19]. As $z \rightarrow +\infty$, $E_\alpha(z) \sim \frac{1}{\alpha} e^{z^{1/\alpha}}$, which resembles the classical case $\alpha = 1$ ($E_1(z) = e^z$). Meanwhile, as $z \rightarrow -\infty$,

$$E_\alpha(z) \sim -\frac{z^{-1}}{\Gamma(1-\alpha)},$$

so that $u_\lambda(t) \sim Ct^{-\alpha}$, $t \rightarrow \infty$. Here and below C denotes various positive constants. This slow decay property is an origin of a large variety of applications of fractional differential equations.

In the distributed order case, where k is given by (2.13) with $\mu(0) \neq 0$, we have a logarithmic decay

$$u_\lambda(t) \sim C(\log t)^{-1}, \quad t \rightarrow \infty.$$

A more complicated choice of μ (or a more general measure instead of $\mu d\alpha$) leads to a diversity of possible decay patterns.

An answer to the above questions regarding conditions upon k guaranteeing (A) and (B) was given in [25]. The sufficient conditions are as follows. The Laplace transform

$$\mathcal{K}(p) = \int_0^\infty e^{-pt} k(t) dt$$

should be a Stieltjes function (or, equivalently, the function $\mathcal{L}(p) = p\mathcal{K}(p)$ should be a complete Bernstein function),

$$\begin{aligned} \mathcal{K}(p) &\rightarrow \infty, \text{ as } p \rightarrow 0; & \mathcal{K}(p) &\rightarrow 0, \text{ as } p \rightarrow \infty; \\ \mathcal{L}(p) &\rightarrow 0, \text{ as } p \rightarrow 0; & \mathcal{L}(p) &\rightarrow \infty, \text{ as } p \rightarrow \infty. \end{aligned}$$

Under these conditions, $\mathcal{L}(p)$ and its analytic continuation admit an integral representation [37]

$$\mathcal{L}(p) = \int_0^\infty \frac{p}{p+t} \sigma(dt) \tag{2.16}$$

where σ is a Borel measure on $[0, \infty)$, such that $\int_0^\infty (1+t)^{-1} \sigma(dt) < \infty$.

2.3. Solutions of the evolution equations

$$\frac{\partial u_1(t, x)}{\partial t} = (A_x u_1)(t, x), \tag{2.17}$$

$$(\mathbb{D}_{(k)} u_{(k)})(t, x) = (A_x u_{(k)})(t, x), \tag{2.18}$$

with the same operator A_x acting in the spatial variables and the same initial conditions

$$u_1(0, x) = \xi(x), \quad u_{(k)}(0, x) = \xi(x),$$

typically satisfy the subordination identity, that is there exists a nonnegative function $G(s, t)$, $s, t > 0$, such that $\int_0^\infty G(s, t) ds = 1$ and

$$u_{(k)}(t, x) = \int_0^\infty G(s, t) u_1(s, x) ds. \tag{2.19}$$

The appropriate notions of solutions of (2.17) and (2.18) depend on the specific setting and were explained in [25] for the case where A is the Laplace operator on \mathbb{R}^n , in [3, 4, 5] (for special classes of functions k) in the setting with abstract semigroup generators, in [34] for abstract Volterra equations. There is also a probabilistic interpretation of subordination identities (see, for example, [26, 35]). In the models of statistical dynamics considered below, we will deal with a subordination of measure flows that will give a weak solution to corresponding fractional equation.

In the above relation (2.19), the subordination kernel does not depend on A and can be found as follows [25]. Consider the function

$$g(s, p) = \mathcal{K}(p)e^{-s\mathcal{L}(p)}, \quad s > 0, p > 0.$$

The function $p \mapsto e^{-s\mathcal{L}(p)}$ is completely monotone (see conditions for the complete monotonicity in Chapter 13 of [14]). By Bernstein's theorem, for each $s \geq 0$, there exists such a probability measure $\mu_s(d\tau)$ that

$$e^{-s\mathcal{L}(p)} = \int_0^\infty e^{-p\tau} \mu_s(d\tau).$$

The family of measures $\{\mu_s\}$ is weakly continuous in s . Then we set

$$G(s, t) = \int_0^t k(t - \tau) \mu_s(d\tau). \quad (2.20)$$

We can find the Laplace transform of G in the variable t :

$$g(s, p) = \int_0^\infty e^{-pt} G(s, t) dt. \quad (2.21)$$

3 Statistical dynamics and fractional kinetics

We will consider Markov dynamics of interacting particle systems in \mathbb{R}^d . The phase space of such systems is the configuration space over the space \mathbb{R}^d which consists of all locally finite subsets (configurations) of \mathbb{R}^d , namely,

$$\Gamma = \Gamma(\mathbb{R}^d) := \{\gamma \subset \mathbb{R}^d \mid |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_b(\mathbb{R}^d)\}, \quad (3.1)$$

where $\mathcal{B}_b(\mathbb{R}^d)$ denotes the family of bounded Borel subsets from \mathbb{R}^d . The space Γ is equipped with the vague topology, i.e., the minimal topology for which all mappings $\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} f(x) \in \mathbb{R}$ are continuous for any continuous function f on \mathbb{R}^d with compact support. Note that the summation in $\sum_{x \in \gamma} f(x)$ is taken over only finitely many points of γ belonging to the support of f . It was shown in [28] that with the vague topology Γ may be metrizable and it becomes a Polish space (i.e., a complete separable metric space). Corresponding to this topology, the Borel σ -algebra $\mathcal{B}(\Gamma)$ is the smallest σ -algebra for which all mappings

$$\Gamma \ni \gamma \mapsto |\gamma_\Lambda| \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$$

are measurable for any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. Here $\gamma_\Lambda := \gamma \cap \Lambda$, and $|\cdot|$ the cardinality of a finite set. Together with Γ , it is useful to introduce a space Γ_0 which consists of all finite configurations in \mathbb{R}^d [27].

A description of each particular model includes a heuristic Markov generator L defined on functions over the configuration space Γ of the system. We assume that the initial distribution (the state of particles) in our system is a probability measure $\mu_0 \in \mathcal{M}^1(\Gamma)$ with corresponding sequence of correlation functions $\varkappa_0 = (\varkappa_0^{(n)})_{n=0}^\infty$, see e.g. [27]. The distribution of particles at time $t > 0$ is the measure $\mu_t \in \mathcal{M}^1(\Gamma)$, and $\varkappa_t = (\varkappa_t^{(n)})_{n=0}^\infty$ its correlation functions. If the evolution of states $(\mu_t)_{t \geq 0}$ is determined by a heuristic Markov generator L , then μ_t is the solution of the forward Kolmogorov equation (or Fokker-Plank equation (FPE)),

$$\begin{cases} \frac{\partial \mu_t}{\partial t} &= L^* \mu_t \\ \mu_t|_{t=0} &= \mu_0, \end{cases} \quad (3.2)$$

where L^* is the adjoint operator. In terms of the time-dependent correlation functions $(\varkappa_t)_{t \geq 0}$ corresponding to $(\mu_t)_{t \geq 0}$, the FPE may be rewritten as an infinite system of evolution equations

$$\begin{cases} \frac{\partial \varkappa_t^{(n)}}{\partial t} &= (L^\Delta \varkappa_t)^{(n)} \\ \varkappa_t^{(n)}|_{t=0} &= \varkappa_0^{(n)}, \quad n \geq 0, \end{cases} \quad (3.3)$$

where L^Δ is the image of L^* in a space of vector-functions $\varkappa_t = (\varkappa_t^{(n)})_{n=0}^\infty$. In applications to concrete models, the expression for the operator L^Δ is obtained from the operator L via combinatoric calculations (cf. [27]).

The evolution equation (3.3) is nothing but a hierarchical system of equations corresponding to the Markov generator L . This system is the analogue of the BBGKY-hierarchy of the Hamiltonian dynamics [6].

Our interest now turns to Vlasov-type scaling of stochastic dynamics for the IPS in a continuum. This scaling leads to so-called kinetic description of the considered model. In the language of theoretical physics we are dealing with a mean-field type scaling which is adopted to preserve the spatial structure. In addition, this scaling will lead to the limiting hierarchy, which possesses a chaos preservation property. In other words, if the initial distribution is Poisson (non-homogeneous) then the time evolution of states will maintain this property. We refer to [15] for a general approach, concrete examples, and additional references.

There exists a standard procedure for deriving Vlasov scaling from the generator in (3.3). The specific type of scaling is dictated by the model in question. The process leading from L^Δ to the rescaled Vlasov operator L_V^Δ produces a non-Markovian generator L_V since it lacks the positivity-preserving property. Therefore instead of (3.2) we consider the following kinetic FPE,

$$\begin{cases} \frac{\partial \mu_t}{\partial t} &= L_V^* \mu_t \\ \mu_t|_{t=0} &= \mu_0, \end{cases} \quad (3.4)$$

and observe that if the initial distribution satisfies $\mu_0 = \pi_{\rho_0}$, then the solution is of the same type, i.e., $\mu_t = \pi_{\rho_t}$.

In terms of correlation functions, the kinetic FPE (3.4) gives rise to the following Vlasov-type hierarchical chain (Vlasov hierarchy)

$$\begin{cases} \frac{\partial \varkappa_t^{(n)}}{\partial t} &= (L_V^\Delta \varkappa_t)^{(n)} \\ \varkappa_t^{(n)}|_{t=0} &= \varkappa_0^{(n)}, \quad n \geq 0. \end{cases} \quad (3.5)$$

Let us consider so-called Lebesgue-Poisson exponents

$$\varkappa_0(\eta) = e_\lambda(\rho_0, \eta) = \prod_{x \in \eta} \rho_0(x)$$

as the initial condition. Such correlation functions correspond to Poisson measures π_{ρ_0} on Γ with the density ρ_0 . The scaling L_V^Δ should be such that the dynamics $\varkappa_0 \mapsto \varkappa_t$ preserves this structure, or more precisely, \varkappa_t should be of the same type

$$\varkappa_t(\eta) = e_\lambda(\rho_t, \eta) = \prod_{x \in \eta} \rho_t(x), \quad \eta \in \Gamma_0. \quad (3.6)$$

Relation (3.6) is known as the *chaos propagation property* of the Vlasov hierarchy. It turns out that equation (3.6) implies, in general, a non-linear differential equation

$$\frac{\partial \rho_t(x)}{\partial t} = \vartheta(\rho_t)(x), \quad x \in \mathbb{R}^d, \quad (3.7)$$

for ρ_t , which is called the *Vlasov-type kinetic equation*.

In general, if one does not start with a Poisson measure, the solution will leave the space $\mathcal{M}^1(\Gamma)$. To have a bigger class of initial measures, we may consider the cone inside $\mathcal{M}^1(\Gamma)$ generated by convex combinations of Poisson measures, denoted by $\mathbb{P}(\Gamma)$.

Below we discuss the concept of a fractional Fokker-Planck equation and the related fractional statistical dynamics, which is still an evolution in the space of probability measures on the configuration space. The mesoscopic scaling of this evolutions leads to a fractional kinetic FPE. A subordination principle provides for the representation of the solution to this equation as a flow of measures that is a transformation of a Poisson flow for the initial kinetic FPE.

We will introduce the fractional statistical dynamics for a given Markov generator L by changing the time derivative in the FPE to $\mathbb{D}_{(k)}$. The resulting fractional Fokker-Planck dynamics (if it exists) will act in the space of states on Γ , i.e., it will preserve probability measures on Γ . The fractional Fokker-Planck equation (FFPE)

$$\begin{cases} \mathbb{D}_{(k)} \mu_t^{(k)} &= L^* \mu_t^{(k)} \\ \mu_t^{(k)}|_{t=0} &= \mu_0^{(k)}. \end{cases} \quad (\text{FFPE})$$

describes a dynamical system with memory in the space of measures on Γ . The corresponding evolution no longer has the semigroup property. However, if the solution μ_t of equation (3.4) exists, then the subordination principle described above shall give for the solution of (FFPE)

$$\mu_t^{(k)} = \int_0^\infty G(s, t) \mu_s ds. \quad (3.8)$$

An application of the subordination principle may be justified in many particular models where the evolution of correlation functions may be constructed by means a C_0 -semigroup in a proper Banach space. In general, the subordination formula may be considered as a rule for the transformation of Markov dynamics to fractional ones.

It is easy to see that $\mu_t^{(k)}$ is a measure. The FFPE equation may be written in terms of time-dependent correlation functions as an infinite system of evolution equations, the so-called *hierarchical chain*:

$$\begin{cases} \mathbb{D}_{(k)} \mathfrak{X}_{(k),t}^{(n)} &= (L^\Delta \mathfrak{X}_{(k),t})^{(n)} \\ \mathfrak{X}_{(k),t}^{(n)}|_{t=0} &= \mathfrak{X}_{(k),0}^{(n)}, \quad n \geq 0. \end{cases}$$

The evolution of the correlation functions should also be given by the subordination principle. More precisely, if the solution \mathfrak{X}_t of equation (3.5) exists and satisfy certain exponential growth bound (as in examples considered below), then we have

$$\mathfrak{X}_{(k),t} = \int_0^\infty G(s, t) \mathfrak{X}_s ds.$$

As in the case of Markov statistical dynamics addressed above, we may consider Vlasov-type scaling in the framework of the FFPE. We know that the kinetic statistical dynamics for a Poisson initial state π_{ρ_0} is given by a flow of Poisson measures

$$\mathbb{R}_+ \ni t \mapsto \mu_t = \pi_{\rho_t} \in \mathcal{M}^1(\Gamma),$$

where ρ_t is the solution to the corresponding Vlasov kinetic equation. Then the fractional kinetic dynamics of states may be obtained as the subordination of this flow. Specifically, we consider the subordinated flow

$$\mu_t^{(k)} := \int_0^\infty G(s, t) \mu_s ds.$$

The family of measures $\mu_t^{(k)}$ is no longer a Poisson flow. We would like to analyze the properties of these subordinated flows to distinguish the effects of fractional evolution. It is reasonable to study the properties of subordinated flows from a more general point of view when the evolution of densities $\rho_t(x)$ is not necessarily related to a particular Vlasov-type kinetic equation.

4 Subordination and intermittency

As we already discussed, for the fractional kinetic hierarchies the correlation functions have the following representation

$$\mathfrak{X}_t^{(n)}(x_1, \dots, x_n) = \int_0^\infty G(s, t) \prod_{j=1}^n \mathfrak{X}_s^{(1)}(x_j) ds.$$

Let us consider a model situation $\mathfrak{X}_s^{(1)}(x) \equiv e^{\beta s}$, $\beta > 0$, so that

$$\mathfrak{X}_t^{(n)} = \int_0^\infty G(s, t) e^{n\beta s} ds. \tag{4.1}$$

This situation is realized, in particular, in the kinetic limit of the spatial contact model in the supercritical regime, see [15, 29]. The existence of the integral in (4.1) will be proved later. We will study an intermittency property of the solution to the kinetic hierarchy in the considered case. For a general discussion concerning the notion of intermittency see [7, 8]. Note that the intermittency property for random fields are formulated in terms of their moments. But in the case of random point processes we are dealing with there is an alternative possibility to reformulate this property in terms of correlation functions, see [10]. For our case, the intermittency property means that for each $n > 1$, and the natural numbers m_1, \dots, m_k , such that $m_1 + \dots + m_k = n$,

$$\frac{\mathcal{Z}_t^{(n)}}{\prod_{j=1}^k \mathcal{Z}_t^{(m_j)}} \longrightarrow \infty, \quad \text{as } t \rightarrow \infty. \quad (4.2)$$

Theorem 4.1. *The intermittency property (4.2) is fulfilled, if*

$$\int_1^\infty \frac{ds}{s\mathcal{L}(s)} < \infty. \quad (4.3)$$

Proof. Let us consider the function

$$A(t, z) = \int_0^\infty e^{zs} G(s, t) ds, \quad t > 0, z > 0. \quad (4.4)$$

By the Fubini-Tonelli theorem, the existence of the integral in (4.4) for almost all $t > 0$ follows from the absolute convergence of the repeated integral

$$\int_0^\infty e^{zs} ds \int_0^\infty e^{-pt} G(s, t) dt = \int_0^\infty e^{zs} g(s, p) ds = \frac{\mathcal{K}(p)}{\mathcal{L}(p) - z}$$

where $p > 0$ is such that $\mathcal{L}(p) > z$.

Therefore the function $A(t, z)$ exists almost everywhere and is locally integrable in t for each fixed z . Its Laplace transform

$$\tilde{A}(p, z) = \int_0^\infty e^{-pt} A(t, z) dt$$

is defined for $\mathcal{L}(p) > z$. For such values of p ,

$$\tilde{A}(p, z) = \frac{\mathcal{K}(p)}{\mathcal{L}(p) - z}.$$

Since $\mathcal{L}(p)$ is a Bernstein function, that is its derivative is completely monotone, by Bernstein's theorem, $\mathcal{L}'(p) \neq 0$ for all $p > 0$ ($\mathcal{L}(p)$ is not a constant function by our assumptions). Therefore \mathcal{L} is strictly monotone. For each $z > 0$, there exists a unique $p_0 = p_0(z)$, such that

$\mathcal{L}(p_0) = z$. The condition $\mathcal{L}(p) > z$ is equivalent to the inequality $p > p_0(z)$. Note that, by virtue of (2.16), $\mathcal{L}(p) \neq z$ for any nonreal p , since $\mathcal{L}(p)$ preserves the open upper and lower half-planes.

It follows from (2.16) that $\tilde{A}(p, z)$ is holomorphic in p on any sector $p_0 + \Sigma_{\rho + \frac{\pi}{2}}$, $0 < \rho < \frac{\pi}{2}$. Here $\Sigma_\delta = \{re^{i\theta} : r > 0, -\delta < \theta < \delta\}$, $\delta > 0$. In addition,

$$\sup_{p \in p_0 + \Sigma_{\rho + \frac{\pi}{2}}} |(p - p_0)\tilde{A}(p, z)| < \infty.$$

By Theorem 2.6.1 from [1], the function $A(t, z)$ is actually holomorphic in t on any sector Σ_v , $0 < v < \rho$, and

$$\sup_{t \in \Sigma_v} |e^{-p_0 t} A(t, z)| < \infty.$$

Let us rewrite \tilde{A} as follows:

$$\tilde{A}(p, z) = \frac{1}{p} \left(1 + \frac{z}{\mathcal{L}(p) - z} \right).$$

This implies the relation $A(t, z) = 1 + B(t, z)$ where B has the Laplace transform

$$\tilde{B}(p, z) = \frac{z}{p} \cdot \frac{1}{\mathcal{L}(p) - z}.$$

It is known that the complete Bernstein function \mathcal{L} satisfies, outside the negative real semi-axis, the inequality

$$\sqrt{\frac{1 + \cos \varphi}{2}} \mathcal{L}(|p|) \leq |\mathcal{L}(p)| \leq \sqrt{\frac{2}{1 + \cos \varphi}} \mathcal{L}(|p|), \quad \varphi = \arg p \quad (4.5)$$

(see Proposition 2.4 in [2]). In particular, on any vertical line $p = \gamma + i\lambda$, $\gamma > p_0$, we have $|\mathcal{L}(p)| \geq \frac{1}{\sqrt{2}} \mathcal{L}(|p|)$. Together with the assumption (4.3), this implies the absolute integrability on such a line of the function $\tilde{B}(p, z)$, as well as the fact that $\tilde{B}(p, z) \rightarrow 0$, as $p \rightarrow \infty$ in the half-plane $\operatorname{Re} p > p_0$. Having these properties (see Theorem 28.2 in [12]), we can write the inversion formula

$$A(t, z) = 1 + \frac{z}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt} \frac{dp}{p(\mathcal{L}(p) - z)}, \quad \gamma > p_0. \quad (4.6)$$

We use (4.6) to study the asymptotics of $A(t, z)$ for a fixed z , as $t \rightarrow \infty$. Denote

$$I_0(t, z) = 1 + \frac{z}{2\pi i} \int_{r - i\infty}^{r + i\infty} e^{pt} \frac{dp}{p(\mathcal{L}(p) - z)}$$

where $0 < r < p_0$. We have

$$|I_0(t, z)| \leq 1 + Ce^{rt} \left| \int_{-\infty}^{\infty} e^{i\lambda t} \frac{d\lambda}{(r + i\lambda)(\mathcal{L}(r + i\lambda) - z)} \right| = o(e^{rt}), \quad t \rightarrow \infty, \quad (4.7)$$

due to (4.3), (4.5) and the Riemann-Lebesgue theorem.

On the other hand,

$$A(t, z) - I_0(t, z) = \frac{z}{2\pi i} \left(\int_{\Gamma_+} + \int_{\Gamma_0} + \int_{\Gamma_-} \right) e^{pt} \frac{dp}{p(\mathcal{L}(p) - z)}$$

where the contour Γ_+ consists of the vertical rays $\{\operatorname{Re} p = r, \operatorname{Im} p \geq R\}$, $\{\operatorname{Re} p = \gamma, \operatorname{Im} p \geq R\}$, and the horizontal segment $\{r \leq \operatorname{Re} p \leq \gamma, \operatorname{Im} p = R\}$ ($R > 0$), Γ_- is a mirror reflection of Γ_+ with respect to the real axis, Γ_0 is the finite rectangular contour consisting of the vertical segments $\{\operatorname{Re} p = r, |\operatorname{Im} p| \leq R\}$, $\{\operatorname{Re} p = \gamma, |\operatorname{Im} p| \leq R\}$, and the horizontal segments $\{r \leq \operatorname{Re} p \leq \gamma, \operatorname{Im} p = \pm R\}$.

We have

$$\int_{\Gamma_+} e^{pt} \frac{dp}{p(\mathcal{L}(p) - z)} = 0.$$

That follows from the Cauchy theorem, absolute integrability of the integrand on the vertical rays, and the estimate of the integral over the horizontal segment $\Pi_h = \{r \leq \operatorname{Re} p \leq \gamma, \operatorname{Im} p = h\}$ ($h > R$):

$$\left| \int_{\Pi_h} e^{pt} \frac{dp}{p(\mathcal{L}(p) - z)} \right| \leq Ch^{-1} \rightarrow 0,$$

as $h \rightarrow \infty$. Similarly,

$$\int_{\Gamma_-} e^{pt} \frac{dp}{p(\mathcal{L}(p) - z)} = 0.$$

Since $\mathcal{L}'(p_0) \neq 0$, there exists a complex neighborhood U of $z = \mathcal{L}(p_0)$, where the function \mathcal{L} possesses a single-valued holomorphic inverse function $p = \psi(w)$, so that $\mathcal{L}(\psi(w)) = w$ and $p_0 = \psi(z)$. Up to now, the numbers r, γ, R were arbitrary. Choose R and $\gamma - r$ so small that the curvilinear rectangle $\mathcal{L}(\Gamma_0)$ lies within U . Making the change of variables $p = \psi(w)$ and using the Cauchy formula we find that

$$\begin{aligned} \frac{z}{2\pi i} \int_{\Gamma_0} e^{pt} \frac{dp}{p(\mathcal{L}(p) - z)} &= \frac{z}{2\pi i} \int_{\mathcal{L}(\Gamma_0)} e^{\psi(w)t} \frac{1}{\mathcal{L}'(\psi(w))\psi(w)} \cdot \frac{dw}{w - z} = \frac{z}{\mathcal{L}'(\psi(z))\psi(z)} e^{\psi(z)t} \\ &= \frac{z}{\mathcal{L}'(p_0(z))p_0(z)} e^{p_0(z)t}. \end{aligned}$$

Together with (4.7), this yields the asymptotic relation

$$A(t, z) = \frac{z}{\mathcal{L}'(p_0(z))p_0(z)} e^{p_0(z)t} + o(e^{p_0(z)t}), \quad t \rightarrow \infty. \quad (4.8)$$

In the next lemma we use the duality of sub- and superadditivity [33].

Lemma 4.1. *The function $p_0(z)$, $z > 0$, is strictly superadditive; in particular,*

$$p_0(n\beta) > \sum_{j=1}^k p_0(m_j\beta)$$

for $n = \sum_{j=1}^k m_j$, $\beta > 0$.

Proof. It is sufficient to prove that

$$p_0(x+y) > p_0(x) + p_0(y) \quad \text{for any } x, y > 0. \quad (4.9)$$

First of all, \mathcal{L} is strictly subadditive, that is

$$\mathcal{L}(a+b) < \mathcal{L}(a) + \mathcal{L}(b), \quad a, b > 0. \quad (4.10)$$

This follows from the integral representation (2.16) and the elementary identity

$$\frac{a}{a+t} + \frac{b}{b+t} - \frac{a+b}{a+b+t} = \frac{a^2b + 2abt + ab^2}{(a+t)(b+t)(a+b+t)}$$

We get from (4.10) and the strict monotonicity of p_0 that

$$a+b < p_0(\mathcal{L}(a) + \mathcal{L}(b)).$$

By our assumptions, \mathcal{L} maps bijectively the semi-axis $[0, \infty)$ onto itself. Choosing a, b in such a way that $\mathcal{L}(a) = x$, $\mathcal{L}(b) = y$, so that $a = p_0(x)$, $b = p_0(y)$, we obtain (4.9). \blacksquare

The asymptotic relation (4.3) follows from (4.8) and Lemma 4.1. \blacksquare

Examples. 1). For the Caputo-Djrbashian fractional derivative $\mathbb{D}_t^{(\alpha)}$, $0 < \alpha < 1$, we have $\mathcal{L}(p) = p^\alpha$, so that (4.3) is satisfied.

Note that [3] in this case $A(t, z) = E_\alpha(z t^\alpha)$ where E_α is the Mittag-Leffler function, and the asymptotic relation (4.8) gives actually the principal term of the asymptotics of E_α . However our proof is different from the well-known proof of the latter (see [11, 19]).

2). Consider a distributed order derivative with a continuous weight function μ , that is

$$\mathbb{D}^{(\mu)}\varphi(t) = \int_0^1 (\mathbb{D}^{(\alpha)}\varphi)(t) \mu(\alpha) d\alpha.$$

In this case (see [23]),

$$k(s) = \int_0^1 \frac{s^{-\alpha}}{\Gamma(1-\alpha)} \mu(\alpha) d\alpha, \quad \mathcal{L}(p) = \int_0^1 p^\alpha \mu(\alpha) d\alpha.$$

It is proved in [23] that, if $\mu \in C^2[0, 1]$, then

$$\mathcal{L}(p) = \frac{\mu(1)p}{\log p} + O(p|\log p|^{-2}), \quad p \rightarrow \infty.$$

Therefore (4.3) is satisfied, if $\mu(1) \neq 0$. See [23] for an investigation of the case where $\mu(1) = 0$.

In the model with decaying correlation functions, it is assumed that

$$\varkappa_t^{(1)} = e^{-\beta t}, \quad \beta > 0. \quad (4.11)$$

This situation is realized in the contact model in subcritical regime [29].

Theorem 4.2. *The intermittency property (4.2) is fulfilled in the case (4.11), if*

$$\mathcal{K}(p) \sim p^{-\gamma} Q\left(\frac{1}{p}\right), \quad p \rightarrow 0, \quad (4.12)$$

where $0 \leq \gamma \leq 1$, Q is a slowly varying function [14, 36].

Proof. Consider the function

$$A(t, -z) = \int_0^\infty e^{-zs} G(s, t) ds, \quad t > 0, z > 0.$$

The existence and boundedness (≤ 1) of this function is obvious. As in the above case, it is in fact analytic in t . Its Laplace transform

$$\tilde{A}(p, -z) = \frac{\mathcal{K}(p)}{\mathcal{L}(p) + z}$$

is a Stieltjes function in the variable p because $p\tilde{A}(p, -z) = \frac{\mathcal{L}(p)}{\mathcal{L}(p) + z}$ is a complete Bernstein function as a composition of the functions \mathcal{L} and $p \mapsto \frac{p}{p+z}$ belonging to this class.

Under our assumptions,

$$\tilde{A}(p, -z) = \int_0^\infty \frac{\sigma(dr)}{p+r},$$

where σ is a Borel measure, $\int_0^\infty \frac{\sigma(dr)}{1+r} < \infty$. To simplify notations, we will write temporarily $h(p)$ instead of $\tilde{A}(p, -z)$ (with a fixed z).

Lemma 4.2. *For each $n \geq 1$, $p > 0$,*

$$\left| \frac{h^{(n+1)}(p)}{h^{(n)}(p)} \right| \leq \frac{n+1}{p}. \quad (4.13)$$

Proof. We have

$$h^{(n)}(p) = (-1)^n n! \int_0^\infty \frac{\sigma(dr)}{(p+r)^{n+1}}, \quad n = 1, 2, \dots,$$

so that

$$|h^{(n+1)}(p)| = \int_0^\infty \frac{(n+1)!}{(p+r)^{n+2}} \sigma(dr) = \int_0^\infty \frac{n!}{(p+r)^{n+1}} \cdot \frac{n+1}{r+p} \sigma(dr) \leq \frac{n+1}{p} |h^{(n)}(p)|,$$

which implies (4.13). \blacksquare

A similar inequality was proved for complete Bernstein functions in [21] (Lemma 3.9.34).

Proof of Theorem 2 (continued). By the Post-Widder formula (see, for example, Theorem 3.8.6 in [21]), for a fixed z ,

$$A(t, -z) = \lim_{n \rightarrow \infty} H_n(t), \quad t > 0,$$

where

$$H_n(t) = \frac{(-1)^n}{n!} h^{(n)} \left(\frac{n}{t} \right) \left(\frac{n}{t} \right)^{n+1}.$$

Each function H_n is non-increasing. Indeed,

$$\begin{aligned} H'_n(t) &= \frac{(-1)^n}{n!} \left(-\frac{n}{t^2} \right) h^{(n+1)} \left(\frac{n}{t} \right) \left(\frac{n}{t} \right)^{n+1} + \frac{(-1)^n}{n!} h^{(n)} \left(\frac{n}{t} \right) (n+1) \left(-\frac{n}{t^2} \right) \left(\frac{n}{t} \right)^n \\ &= \frac{(-1)^n}{n!} \left(\frac{n}{t} \right)^{n+1} \left(-\frac{n}{t^2} \right) h^{(n)} \left(\frac{n}{t} \right) \left[\frac{h^{(n+1)} \left(\frac{n}{t} \right)}{h^{(n)} \left(\frac{n}{t} \right)} + \frac{(n+1)t}{n} \right]. \end{aligned}$$

By (4.13), the expression in square brackets is nonnegative. Since a Stieltjes function is completely monotone, we find that $H'_n(t) \leq 0$ for each n , that is $A(t, -z)$ is non-increasing in the variable t .

By the Karamata-Feller Tauberian theorem [14], the asymptotics (4.12) and the above monotonicity of A imply the asymptotics

$$A(t, -z) \sim \frac{1}{\Gamma(\gamma)z} t^{\gamma-1} Q(t), \quad t \rightarrow \infty.$$

In particular, $\varkappa_t^{(n)} = A(t, -n\beta)$, so that

$$\frac{\varkappa_t^{(n)}}{\prod_{j=1}^k \varkappa_t^{(m_j)}} \sim \text{const} \cdot e^{n\beta t} t^{\gamma-1} Q(t) \rightarrow \infty,$$

as $t \rightarrow \infty$. \blacksquare

Examples. 1) In the case of the Caputo-Djrbashian fractional derivative $\mathbb{D}_t^{(\alpha)}$, $0 < \alpha < 1$, we have $\mathcal{K}(p) = p^{\alpha-1}$, and (4.12) is satisfied.

2) For the distributed order derivative with a continuous weight function μ , we have

$$\mathcal{K}(p) \sim p^{-1} \left(\log \frac{1}{p} \right)^{-1} \mu(0), \quad p \rightarrow 0,$$

if $\mu(0) \neq 0$. Thus, in this case (4.12) holds with $\gamma = 1$.

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